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ON A GEOMETRICAL INTERPRETATION OF ENERGY AND MOMENTUM CONSERVATION IN ATOMIC COLLISIONS AND DISINTEGRATION PROCESSES

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CONTENTS

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INTRODUCTION

In every collision and disintegration process we have to do with two kinds of relations, *viz.*, those which result from pure conservation laws and are valid both in classical and in quantum theory, and others which for the different processes allowed by the conservation laws give the probabilities resulting from quantum-mechanical considerations.

In almost all collision and disintegration processes one or two particles are present in the initial state; it is not necessary to treat separately the case of one particle (spontaneous decay) since it is a special case of a process with two initial particles, one of which has a vanishing mass and momentum. The final state comprises two or more particles. In the following, however, we shall consider processes with two particles only in the final state; a certain extension to the case of three particles will be given in the last section.

The present paper deals only with relations that can be derived from the conservation laws for energy and momentum. The conservation laws are treated in the relativistic form, and only for some of the applications do we go over to the non-relativistic domain. The conservation laws allow an interpretation, by means of a simple geometrical picture, from which it is possible, for every collision or disintegration process, to obtain a qualitative and quantitative survey regarding the possible momenta and energies of the particles in the final state.

For the special case of elastic collisions, this picture was given by LANDAU and LIFSCHITZ [1], but it can easily be extended to all kinds of collision and disintegration processes[1)].

1) Such a picture was also given by WATAGHIN [2] in the discussion of meson production by collision of nucleons.

As will be shown in § 1, the geometrical picture in question consists of a certain prolate ellipsoid of revolution with the major axis in the direction of the total momentum of the system, together with two points, A_1 and A_2, lying on its major axis at a distance apart equal to the total momentum of the system (Fig. 1). This ellipsoid will be called the momentum ellipsoid. If A_1 is taken as the origin of the momentum of one of the two particles in the final state (denoted as particle 1), all possible end points B of this momentum vector lie on the surface of the momentum ellipsoid. The corresponding momentum of particle 2 in the final state will then be given by the vector from B to A_2.

The parameters describing the momentum ellipsoid (e. g., the semi-minor axis and the eccentricity) are easily found from the total energy and the total momentum of the system and the rest masses m_1 and m_2 of the particles 1 and 2, respectively. For a given ellipsoid, these two masses, m_1 and m_2, determine the positions, relative to the ellipsoid, of the corresponding points A_1 and A_2, respectively. The point A_1 may lie inside or outside the momentum ellipsoid or on its surface; this is also true for the point A_2. Consequently, from our geometrical picture, there are several different possibilities regarding the directions of flight of the outcoming particles.

In § 2, the useful relations between corresponding quantities in the laboratory system and the mass centre frame of reference (m.c.f.) are derived from the geometrical properties of the momentum ellipsoid. Directly, we find formulae for the momentum and the energy values of a particle as functions of the angle ϑ between the direction of flight of the particle and the total momentum of the system. Moreover the relation between the angular distributions of the outcoming particles in the m.c.f. and in the laboratory system is derived. Especially simple is the connection between the angular distribution in the m.c.f. and the energy distribution in the laboratory system.

In all practical applications, we can assume that one of the particles in the initial state is at rest; the elements of the geometrical picture for this case are given in § 3. In the following four sections (§ 4 — § 7), the geometrical picture is applied to the discussion of some physically important cases.

Elastic collisions are discussed in § 4. In this case, point A_2, referring to that particle which is at rest before the collision, lies on the surface of the momentum ellipsoid, while the point A_1 lies inside or outside the ellipsoid, depending upon whether the mass of the incident particle is smaller or larger than the mass of the target particle.

The discussion of nuclear reactions in § 5 is simplified by the fact that, here, we have generally to do with nonrelativistic energies for which the momentum ellipsoid goes over into a sphere. This allows a simple geometrical representation of the square roots of the kinetic energies of the outgoing particles.

Photo-disintegration is treated in § 6. It is characteristic of this process (as of all endothermic processes) that the energy and the possible directions of flight of the disintegration products change rapidly for relatively small variations of energy of the incident photon in the neighbourhood of the threshold energy.

§ 7 deals with the spontaneous decay of a meson into two particles. This problem is of particular interest in connection with the discovery, by Powell and collaborators [3], of the heavy meson. Energy and angular distribution of the secondary particles for a given primary meson energy are calculated.

Finally, in § 8, it is shown that the method can be extended to the case of three particles in the final state and, as an example, is given the geometrical picture for the β-decay of a nucleus.

§ 1. Geometrical Picture of the Conservation Laws.

In any process which results in the presence of only two particles the conservation laws for energy and momentum may be easily interpreted geometrically. We consider, in particular, an initial state in which there are two free particles, *I* and *II*, with rest masses m_I and m_{II} and momenta $\vec{k}_I$ and $\vec{k}_{II}$ (momentum and mass will throughout be measured in energy units, i.e. $\vec{k}$ denotes the momentum in usual units times the velocity of light c, and m is the mass in usual units times c^2). The case of a spontaneous decay with one particle only in the initial state can be described by putting $m_{II} = 0$, $\vec{k}_{II} = 0$. The two free particles *1* and *2*, present in the final state, are not necessarily of the same kind as those in the initial state; let their rest masses be m_1 and m_2 and their momenta $\vec{k}_1$ and $\vec{k}_2$.

Denoting the total energy of the system by E, and the total momentum by $\vec{K}$, we have when considering the initial state

$$E = \sqrt{m_I^2 + k_I^2} + \sqrt{m_{II}^2 + k_{II}^2} \tag{1,1}$$

$$\vec{K} = \vec{k}_I + \vec{k}_{II}, \tag{1,2}$$

and, according to the conservation laws,

$$E = \sqrt{m_1^2 + k_1^2} + \sqrt{m_2^2 + k_2^2} \tag{1,3}$$

$$\vec{K} = \vec{k}_1 + \vec{k}_2. \tag{1,4}$$

For any Lorentz frame of reference, the quantity

$$N = \sqrt{E^2 - K^2} \tag{1,5}$$

is an invariant giving the energy of the system in the m.c.f. The velocity of the mass centre in units of light velocity is given by

$$\vec{\beta} = \frac{\vec{K}}{E}. \tag{1,6}$$

In the initial as well as in the final state, the particles move relative to the m.c.f. in opposite directions and have equal absolute values of momentum. For the final state, the momentum value in the m.c.f., which we shall denote by b, is readily computed from the energy conservation law in the m.c.f., and is equal to

$$b = \frac{1}{2N}\sqrt{\{N^2-(m_1+m_2)^2\}\{N^2-(m_1-m_2)^2\}}. \tag{1,7}$$

The corresponding energies

$$e_1^* = \sqrt{m_1^2+b^2} \quad \text{and} \quad e_2^* = \sqrt{m_2^2+b^2} \tag{1,8}$$

of the particles *1* and *2* in the m.c.f. are then

$$e_1^* = \frac{N}{2}\left(1+\frac{m_1^2-m_2^2}{N^2}\right), \quad e_2^* = \frac{N}{2}\left(1+\frac{m_2^2-m_1^2}{N^2}\right), \tag{1,9}$$

the asterisk referring to the m.c.f.

On performing a Lorentz transformation which brings us back to the laboratory frame, with the z-axis in the direction of the total momentum vector $\vec{K}$, we find the following equation for the components of $\vec{k}_1$:

$$\frac{k_{1x}^2+k_{1y}^2}{b^2}+\frac{(k_{1z}-\alpha_1)^2}{a^2} = 1, \tag{1,10}$$

where

$$a = \frac{b}{\sqrt{1-\beta^2}} \tag{1,11}$$

$$\alpha_1 = \frac{\beta}{\sqrt{1-\beta^2}}\, e_1^*. \tag{1,12}$$

An analogous equation in wich α_1 is replaced by

$$\alpha_2 = \frac{\beta}{\sqrt{1-\beta^2}}\, e_2^* = K-\alpha_1 \tag{1,12 a}$$

holds for the components of the momentum $\vec{k}_2$.

Equation (1,10) represents a prolate ellipsoid of revolution with the symmetry axis in the direction of $\vec{K}$, and with the centre displaced by the distance a_1 in the direction of $\vec{K}$ from the origin A_1 of the $\vec{k}_1$ vector[1]. Any vector from A_1 to an arbitrary point B on the surface of the ellipsoid represents, therefore, a possible momentum of the particle *1* in the final state, and vice versa. On the other side of the centre of the ellipsoid, in a distance α_2 from it, lies the point A_2, and the vector from B to A_2 represents the corresponding momentum $\vec{k}_2$ of the particle *2*. The ellipsoid defined by (1,10) will be called the momentum ellipsoid.

The semi-minor axis of the momentum ellipsoid is equal to the common value b of the momentum of both particles in the m.c.f. as determined by (1,7), whereas the semi-major axis a is given by (1,11). The eccentricity of the ellipsoid is, therefore, equal to the velocity β of the mass centre, i. e.

$$\beta = \frac{f}{a}, \tag{1,13}$$

where

$$f = \sqrt{a^2 - b^2} \tag{1,14}$$

is half the distance between the foci (Fig. 1).

The considerations in this and the following section are equally valid for both particles *1* and *2* and we, thus, denote all quantities referring to one of these particles by the index n ($n = 1, 2$).

The distance α_n can, on account of (1,12), (1,8), (1,13), and (1,11), also be written in the form

$$\alpha_n = \frac{f}{b}\sqrt{b^2 + m_n^2} \tag{1,15}$$

1) According to a remark by Professor O. Klein, the ellipsoid (1,10) can be considered to be the section in $\vec{k}_1$ space of a 4-dimensional ellipsoid of revolution in a $(\vec{k}_1, x_4)$ space. The equation of this 4-dimensional ellipsoid is

$$\sqrt{\vec{k}_1^2 + (x_4 - m_1)^2} + \sqrt{(\vec{k}_1 - \vec{K})^2 + (x_4 - m_2)^2} = E.$$

Putting $x_4 = 0$, we have an equation which follows immediately from (1,3) and (1,4). The foci of this 4-dimensional ellipsoid lie in the $\vec{K}, x_4$-plane and have there the coordinates $(0, m_1)$ and (K, m_2), respectively; the major axis is equal to E.

and it is seen that, for a given momentum ellipsoid, the distance α_n is entirely determined by the corresponding mass m_n. From (1,15) results the construction of A_n given in Fig. 2. Hence, we have always

$$\alpha_n \geq f.$$

Equality between α_n and f is established only when $m_n = 0$.

Regarding the possible directions of the momentum of particle n in the final state, we have to distinguish between two cases, *viz.* $\alpha_n \geq a$ and $\alpha_n < a$. In the first case, the vector $\vec{k}_n$ always forms, with the direction of $\vec{K}$, an acute angle ϑ_n which is smaller than or equal to a certain angle $\vartheta_{n\,\max}$, whereas, in the second case no such limitation exists. To distinguish these two cases we introduce

$$\gamma_n = \frac{\alpha_n}{a}. \qquad (1,16)$$

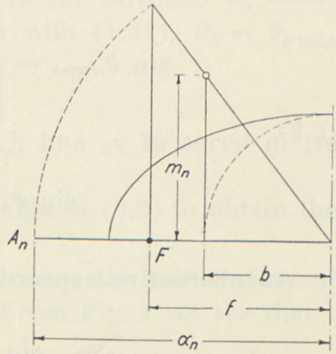

Fig. 2. Geometrical determination of the point A_n (origin of the $\vec{k}_n$ vector) as given by the proportion $\alpha_n : \sqrt{b^2 + m_n^2} = f : b$.

Thus, the first case is characterized by the relation $\gamma_n \geq 1$ and the second case by the relation $\gamma_n < 1$. The quantity γ_n has a simple physical meaning. As seen from (1,11) and (1,12),

$$\gamma_n = \beta \frac{e_n^*}{b}. \qquad (1,17)$$

Taking into account that the velocity β_n^* (in units of light velocity) of the particle n relative to the m.c.f. is equal to

$$\beta_n^* = \frac{b}{e_n^*} \qquad (1,18)$$

we obtain

$$\gamma_n = \frac{\alpha_n}{a} = \frac{\beta}{\beta_n^*}. \qquad (1.19)$$

In the first case, ($\gamma_n \geq 1$), the velocity of the mass centre is, therefore, greater than or equal to the velocity of the particle relative to the m.c.f. and, in the second case, ($\gamma_n < 1$), it is smaller than the velocity of the particle relative to the m.c.f. We have always

$$\gamma_n \geq \beta. \qquad (1,20)$$

Equality exists only when $m_n = 0$.

From Fig. 1 we can easily compute the maximum angle $\vartheta_{n\,\max}$ occurring in the case $\gamma_n \geq 1$:

$$\sin \vartheta_{n\,\max} = \frac{b}{\sqrt{\alpha_n^2 - f^2}}, \qquad \vartheta_{n\,\max} \leq \frac{\pi}{2}$$

or, in terms of γ_n and β,

$$\sin \vartheta_{n\,\max} = \sqrt{\frac{1-\beta^2}{\gamma_n^2 - \beta^2}}. \qquad (1,21)$$

An alternative expression for (1,21) is

$$\sin \vartheta_{n\,\max} = \frac{b^2}{f m_n}. \qquad (1,22)$$

§ 2. Relations between Corresponding Quantities in Laboratory System and m.c.f.

The geometrical interpretation of the energy and momentum conservation leads immediately to a geometrical construction connecting the value and direction of the momentum of a particle in the m.c.f. with the corresponding values in the laboratory frame of reference and shown in Fig. 3.

From Fig. 3 we see that

$$\operatorname{tg} \vartheta_n = \frac{b \sin \vartheta_n^*}{\alpha_n + a \cos \vartheta_n^*} = \sqrt{1-\beta^2}\,\frac{\sin \vartheta_n^*}{\gamma_n + \cos \vartheta_n^*}, \qquad (2,1)$$

the asterisk referring to the m.c.f.

From (2,1) we can also express the angle ϑ_n^* in terms of ϑ_n and obtain

$$\cos\vartheta_n^* = \frac{1}{1-\beta^2\cos^2\vartheta_n}\left\{-\gamma_n\sin^2\vartheta_n \pm (1-\beta^2)\sqrt{1-\frac{\gamma_n^2-\beta^2}{1-\beta^2}\sin^2\vartheta_n}\cos\vartheta_n\right\}. \quad (2,2)$$

The function $\cos\vartheta_n^*$ is, for $\gamma_n > 1$, double-valued since, as it can directly be seen from Fig. 3, the line drawn from A_n at an angle ϑ_n with $\vec{K}$ will, for every angle ϑ_n smaller than $\vartheta_{n\,\mathrm{max}}$, have two points of intersection with the momentum ellipsoid giving, thus, two possible values for ϑ_n^* and two possible momenta in the laboratory system. The plus sign in (2,2) belongs to the greater possible value of momentum in the direction ϑ_n.

As seen from (2,2) in connection with (1,21), $\vartheta_n = \vartheta_{n\,\mathrm{max}}$ corresponds to $\vartheta_{n\,0}^*$ given by

$$\cos\vartheta_{n\,0}^* = -\frac{1}{\gamma_n}. \quad (2,3)$$

For $\gamma_n \leq 1$, we need take only the + sign in (2,2) to obtain the correct value ϑ_n^*.

From (2,2) we can also easily deduce the momentum of particle n going in the direction ϑ_n. From Fig. 3 we see that

$$k_n(\vartheta_n) = a\frac{\gamma_n + \cos\vartheta_n^*}{\cos\vartheta_n} \quad (2,4)$$

so that, according to (2,2),

$$k_n(\vartheta_n) = \frac{p}{1-\beta^2\cos^2\vartheta_n}\left\{\gamma_n\cos\vartheta_n \pm \sqrt{1-\frac{\gamma_n^2-\beta^2}{1-\beta^2}\sin^2\vartheta_n}\right\}, \quad (2,5)$$

where

$$p = b\sqrt{1-\beta^2} \quad (2,6)$$

is the parameter of the ellipsoid. As regards the sign, the same conditions are valid as for (2,2).

The energy $e_n = \sqrt{k_n^2 + m_n^2}$ of the particle n can also be expressed as a function of ϑ_n. The transformation equation for the energy from the m.c.f. to the laboratory system is given by

$$e_n = \frac{e_n^*}{\sqrt{1-\beta^2}} + f\cos\vartheta_n^*. \quad (2,7)$$

From this equation it follows immediately that the energy range of each particle in the final state is given by the distance $2f$ between the foci, i. e.

$$e_{n\,\max} - e_{n\,\min} = 2f \tag{2,8}$$

with

$$e_{n\,\max} = \frac{\alpha_n}{\beta} + f, \quad e_{n\,\min} = \frac{\alpha_n}{\beta} - f. \tag{2,9}$$

Expressing $\cos\vartheta_n^*$ by means of (2,4), we obtain

$$e_n = \sqrt{1-\beta^2}\, e_n^* + \beta\, k_n \cos\vartheta_n = \frac{p}{f}\alpha_n + \beta\, k_n \cos\vartheta_n. \tag{2,10}$$

Apart from the constant $\frac{p}{f}\alpha_n$, the energy of a particle in the final state is, therefore, given by the product of its momentum component in the direction of the total momentum times the eccentricity of the ellipse.

By means of (2,2), we can easily find the connection between the angular distribution of the particles in the laboratory system and in the m.c.f. If we denote by $\sigma_n(\vartheta_n, \varphi)$ and $\sigma_n^*(\vartheta_n^*, \varphi)$ the probability per unit solid angle of finding the particle n in a certain direction in the laboratory system and in the m.c.f., respectively, we have

$$\sigma_n(\vartheta_n, \varphi) = \sigma_n^*(\vartheta_n^*, \varphi)\frac{d\cos\vartheta_n^*}{d\cos\vartheta_n}. \tag{2,11}$$

We have here also, in analogy to (2,2) and (2,5), to distinguish between two cases, *viz.* $\gamma_n > 1$ and $\gamma_n \leq 1$. In the first case, $\cos\vartheta_n$ as a function of $\cos\vartheta_n^*$ has a minimum value $\cos\vartheta_{n\,\max}$. Therefore, $\cos\vartheta_n^*$ as well as $\frac{d\cos\vartheta_n^*}{d\cos\vartheta_n}$ is a double-valued function of $\cos\vartheta_n$, and $\frac{d\cos\vartheta_n^*}{d\cos\vartheta_n}$ becomes infinite for $\vartheta_n = \vartheta_{n\,\max}$. For the two branches (which correspond to different absolute values of momentum), $\frac{d\cos\vartheta_n^*}{d\cos\vartheta_n}$ has different signs. The sign is positive for the particles moving in the direction ϑ_n with a higher value of momentum, and negative for those moving with lower momentum in the same direction. In the latter case, the probability density is given by $|\sigma|$.

For $\gamma_n \leq 1$, $\cos \vartheta_n^*$ is a single-valued function of $\cos \vartheta_n$ and $\frac{d \cos \vartheta_n^*}{d \cos \vartheta_n}$ is single-valued, finite, and positive.

Carrying out the differentiation, we obtain from (2,11), (2,2), and (2,5)

$$\sigma_n (\vartheta_n \varphi) = \pm \sigma^* (\vartheta_n^*, \varphi) \left(\frac{k_n}{b}\right)^2 \frac{1}{\sqrt{1 - \frac{\gamma_n^2 - \beta^2}{1 - \beta^2} \sin^2 \vartheta_n}}, \qquad (2,12)$$

where k_n in terms of ϑ_n is given by (2,5). As regards the sign, the same conditions are valid as in (2,2) and (2,5). We see from (2,12) that, in the case $\gamma_n > 1$, and for a spherical symmetric angular distribution in the m.c.f., the probability density for a given angle ϑ_n is, due to the factor k_n^2, always greater for the particles with the greater than for those with the smaller of the two possible momentum values.

In the case $\gamma_n = 1$ which is, for example, realized in an elastic collision when the particle under consideration is, before the collision, at rest in the laboratory system (cf. § 4), (2,12) is simplified, on account of (2,5), to

$$\sigma_n = \sigma_n^* \frac{4 (1 - \beta^2)}{(1 - \beta^2 \cos^2 \vartheta_n)^2} \cos \vartheta_n .$$

A very simple relation exists between the angular distribution in the m.c.f. and the energy distribution $\sigma_n(e_n)$ per unit energy range in the laboratory system, which follows directly from (2,7). From this equation, we obtain by differentiation

$$d e_n = f d \cos \vartheta_n^* .$$

Since $\sigma_n^*(\vartheta_n^*, \varphi)$ will in general not depend upon the azimuth φ so that $\sigma_n^*(\vartheta_n^*, \varphi) = \sigma_n^*(\vartheta_n^*)$, we get, after integration over the azimuth φ, for $\sigma_n(e_n)$,

$$\sigma_n (e_n) = \frac{2 \pi}{f} \sigma_n^* (\vartheta_n^*), \quad e_{n \min} \leq e_n \leq e_{n \max} ,$$

where we have to express ϑ_n^* in $\sigma_n^*(\vartheta_n^*)$ by e_n in using the formula (2,7):

$$\cos\vartheta_n^* = \frac{1}{f}\left(e_n - \frac{\alpha_n}{\beta}\right).$$

If the angular distribution in the m.c.f. is spherically symmetrical ($\sigma^* = \frac{1}{4\pi}$), the probability $\sigma_n(e_n)$ per unit range of energy is constant and equal to

$$\sigma_n(e_n) = \frac{1}{2f} \qquad e_{n\,\mathrm{min}} \leq e_n \leq e_{n\,\mathrm{max}}$$

which is well known in the non-relativistic case [4].

§ 3. Collisions with One Particle Originally at Rest.

In all cases of practical importance, we may assume that in the laboratory system one of the two particles in the initial state is at rest. In the following applications (§§ 4—8), particle II will always be considered as being at rest ($\vec{k}_{II} = 0$). It then follows from (1,1) and (1,2) that

$$\vec{K} = \vec{k}_I, \qquad E = \sqrt{m_I^2 + K^2} + m_{II} \tag{3,1}$$

and, for the energy N of the system in the m.c.f.,

$$N = \sqrt{M_i^2 + 2\,m_{II}\,W_i}, \tag{3,2}$$

where the index i refers to the initial state of the system, W is the kinetic energy (defined as the total energy minus the rest energy) of the system

$$W_i = \sqrt{m_I^2 + K^2} - m_I, \tag{3,3}$$

and M is the sum of the rest masses of the particles present:

$$M_i = m_I + m_{II}, \tag{3,4}$$

$$M_f = m_1 + m_2, \tag{3,5}$$

the index f referring to the final state of the system.

The reaction energy Q released in the process is then equal to

$$Q = M_i - M_f, \tag{3,6}$$

which is positive for exothermic and negative for endothermic processes. If

$$M = \frac{1}{2}(M_i + M_f) \tag{3,7}$$

is the arithmetical mean of the sum of the rest masses in the initial and the final state, we can write the value b of the semi-minor axis of the momentum ellipsoid in the form

$$b = \sqrt{\frac{(QM + m_{II} W_i)(QM + m_{II} W_i + 2 m_1 m_2)}{M_i^2 + 2 m_{II} W_i}}. \tag{3,8}$$

For endothermic processes ($Q < 0$), we obtain from this equation the threshold energy $W_i = \delta$ from the condition that b must be real,

$$\delta = \frac{M}{m_{II}} |Q| \tag{3,9}$$

and, for this value of W_i we have $b = a = 0$.

Using (3, 6), we get

$$\delta = \frac{M_f^2 - M_i^2}{2 m_{II}}. \tag{3,10}$$

§ 4. Elastic Scattering.

In this particular case, the rest masses of the two particles remain unchanged throughout the process ($m_1 = m_I$, $m_2 = m_{II}$ and, consequently, $M = M_i = M_f$). The momentum ellipsoid is determined by its eccentricity β (= mass centre velocity) computed from (1,6), (3,3), (3,1), and (3,4):

$$\beta = \frac{\sqrt{W_i(2 m_1 + W_i)}}{M + W_i}. \tag{4,1}$$

and the parameter $p = b\sqrt{1 - \beta^2}$, according to (4,1), (3,8), and (3,6), is given by

$$p = m_2 \beta . \tag{4,2}$$

The semi-minor and semi-major axes are then

$$b = \frac{m_2 \beta}{\sqrt{1-\beta^2}}, \quad a = \frac{m_2 \beta}{1-\beta^2}. \tag{4,3}$$

The distances α_1 and α_2 are, according to (1,12), (1,9), (3,2), and (4,3),

$$\alpha_1 = \frac{W_i + \frac{m_1}{m_2} M}{W_i + M} a, \quad \alpha_2 = a . \tag{4,4}$$

From (4,4) and (1,16) we see at once that $\gamma_1 \geq 1$ for $m_1 \geq m_2$ and $\gamma_1 < 1$ for $m_1 < m_2$, while $\gamma_2 = 1$, since the particle which is at rest before the collision has the velocity $\beta_1^* = \beta$ relative to the m.c.f. in the initial as well as in the final state.

The three possible cases, *viz.* $m_1 = m_2$, $m_1 > m_2$, and $m_1 < m_2$, are illustrated in Figs. 4, 5, and 6.

From the geometrical picture we can easily deduce the following results for elastic collisions:

(1) The momentum of the target particle moving after the collision in the direction ϑ_2 is, from (2,5),

$$k_2 = \frac{2 m_2 \beta \cos \vartheta_2}{1 - \beta^2 \cos^2 \vartheta_2}. \tag{4,5}$$

(2) The maximum momentum $k_{2\,\mathrm{max}}$ transferred in an elastic collision to the struck particle is given by the major axis ($= 2a$) of the ellipsoid as determined by (4, 3).

(3) The kinetic energy transferred in the collision to the target particle is, from (2,10), (4,2), and (4,4),

$$w_2 = \beta\, k_2 \cos \vartheta_2 \tag{4,6}$$

and is, therefore, given by the product of the momentum component of the target particle in the primary direction times the eccentricity of the ellipse.

(4) The maximum amount of energy is transferred when $\vartheta_2 = 0$ and is

$$(w_2)_{\max} = 2\, m_2 \frac{\beta^2}{1-\beta^2} = 2f, \tag{4,7}$$

$(w_2)_{\max}$ is, therefore, equal to the distance between the foci, as is also seen directly from (2,8).

(5) The angle ϑ_2 between the motion of the target particle and the primary direction of the incident particle is, from (4,6),

$$\cos\vartheta_2 = \frac{1}{\beta}\sqrt{\frac{w_2}{w_2+2\,m_2}}. \tag{4,8}$$

(6) The range of angles ϑ_2 for the target particle, for a given energy transfer w_2, is from (4,8)

$$\vartheta_2(w_2) < \Theta_2(w_2) \text{ where } \operatorname{ctg}\Theta_2 = \sqrt{\frac{w_2}{2\,m_2}}. \tag{4,9}$$

This limiting angle Θ_2 is independent of the mass and the energy of the incident particle. From (4,9) it follows that the path of a particle which receives a relativistic energy in an elastic collision ($w_2 \gg m_2$) always forms a very small angle with the primary direction. Further, for a given angle ϑ_2, we have $\frac{w_2}{m_2} < 2\operatorname{ctg}^2\vartheta_2$. [5]

(7) The mass m_1 of the incident particle can be determined from the knowledge of its primary momentum K (or its magnetic rigidity) and the direction of flight and momentum of a struck particle of known mass [6]. By means of (1,6) and (4,6) we can determine the total energy of the system $E = \frac{K}{\beta} = K\frac{k_2\cos\vartheta_2}{w_2}$ and, therefore, also m_1 which, according to (3,1), is given by

$$m_1^2 = \left(\frac{K\,k_2\cos\vartheta_2}{w_2} - m_2\right)^2 - K^2.$$

This formula becomes simpler in the case $m_1 \gg m_2$, for instance for collisions of mesons with electrons.

(8) The kinetic energy W_i of the incident particle which makes an elastic collision with another particle of equal mass ($m_1 = m_2 = m$) can be determined solely from the two directions after the collision (from the ellipse in Fig. 4):

$$\frac{W_i}{2m} + 1 = \operatorname{ctg} \vartheta_1 \operatorname{ctg} \vartheta_2. \tag{4,10}$$

For kinetic energies small compared with the rest energy, we have $\vartheta_1 + \vartheta_2 \sim \frac{\pi}{2}$, as is well known for non-relativistic velocities. Formula (4,10) is, therefore, convenient only for the energy determination of fast particles. It has been checked experimentally by CHAMPION [7] for collisions between two electrons.

(9) The angle ψ between the particles after the collision, in the case $m_1 > m_2$, can vary between *0* and $\frac{\pi}{2}$, but for $m_1 = m_2$ there is a lower limit $\psi_{\min}$ for this angle, given by

$$\cos \psi_{\min} = \frac{W_i}{W_i + 4m}.$$

The minimum value is attained for $\vartheta_1 = \vartheta_2$.

(10) The momentum k_1 of a photon of original momentum K, scattered through an angle ϑ_1 by a particle of mass m_2, is given by the standard form of the equation for an ellipse in polar coordinates

$$k_1 = \frac{m_2 \beta}{1 - \beta \cos \vartheta_2} \tag{4,11}$$

$$\beta = \frac{K}{m_2 + K}. \tag{4,12}$$

This equation represents the well-known Compton formula.

The energy determination of a photon by measuring the momentum (or energy) of the Compton electron is often used [8]. From (4,11), (4,12) and (4,6) we have

$$K = \frac{m_2}{\frac{k_2}{w_2} \cos \vartheta_2 - 1}.$$

For $\vartheta_2 = 0$, in which direction the probability for the motion of the electron is highest, we have from the figure

$$K = a + f = \frac{1}{2}(k_2 + w_2).$$

(11) In the case $m_1 > m_2$, the maximum deflection angle $\vartheta_{1\,\max}$ of the incident particle is, from (1,22) and (4,3), given by

$$\sin \vartheta_{1\,\max} = \frac{m_2}{m_1}.$$

§ 5. Nuclear Reactions.

For the application of the geometrical picture of energy-momentum conservation to nuclear reactions, we may confine ourselves to the non-relativistic domain and treat only the case in which the kinetic energy both in the initial and the final state is small compared with the rest energy of any of the particles under consideration. The momentum ellipsoid goes then over into a sphere with the radius, given from (3,8),

$$a = b = \frac{1}{M}\sqrt{2\, m_1 m_2 (m_{II} W_i + MQ)}. \qquad (5,1)$$

The distance α_n is found from (1,12), by putting

$$e_n^* = m_n, \quad \beta = \frac{K}{M},$$

to be equal to

$$\alpha_n = \frac{m_n}{M} K.$$

Also for physical reasons, this formula is clear, α_n now representing that part of the momentum of particle n which comes from the motion of the mass centre, as seen in Fig. 7.

Expressing α_n in terms of the kinetic energy W_i of the incident particle, we obtain

$$\alpha_n = \frac{m_n}{M}\sqrt{2\, m_I W_i}. \qquad (5,2)$$

For the discussion of the different cases, it is convenient to introduce the quantity

$$\delta' = \frac{M}{m_{II}} Q \tag{5,3}$$

which, in the special case of endothermic processes, represents the threshold energy δ with negative sign.

The directions in which particles *1* and *2* in the final state can move are determined by γ_1 and γ_2 which, according to (1,16), (5,1), (5,2), and (5,3), are given by

$$\gamma_1 = \frac{\alpha_1}{a} = \sqrt{\frac{m_1}{m_2}\frac{m_I}{m_{II}}\frac{W_i}{W_i + \delta'}}, \quad \gamma_2 = \frac{\alpha_2}{a} = \sqrt{\frac{m_2}{m_1}\frac{m_I}{m_{II}}\frac{W_i}{W_i + \delta'}}. \tag{5,4}$$

For $\gamma_n \geq 1$ the direction of the outgoing particle n forms with the primary direction an acute angle smaller than or equal to $\vartheta_{n\,\mathrm{max}}$ which, according to (1,21), in the non-relativistic case considered here, is given by

$$\sin \vartheta_{n\,\mathrm{max}} = \frac{1}{\gamma_n}. \tag{5,5}$$

From now on, we shall for convenience ascribe the index *1* to that of the two particles in the final state which has the smaller mass, i. e.

$$m_1 \leq m_2.$$

In almost all nuclear reactions[1] $m_I \leq m_{II}$. Obviously,

$$\sqrt{\frac{W_i}{W_i + \delta'}} < 1\ (> 1)$$

for exothermic (endothermic) reaction and increases (decreases) with increasing W_i. With the aid of these data we can, from (5,4), derive the following results.

Exothermic reactions.

(1) The quantity γ_1 is always smaller than unity; consequently, the direction of motion of the lighter particle, or of both particles if they have equal masses, is not limited.

1 In fact, the only exception is presented by the bombardment of deuterium with α-particles.

(2) The quantity γ_2 is greater than or equal to unity only in reactions in which $m_I > m_1$ (i. e. in (d, p), (d, n), (α, p), and (α, n) reactions) and in which the energy W_i satisfies the condition

$$W_i \geq \frac{m_1}{m_I - m_1} Q.$$

The direction of motion of the recoil nucleus is then limited by (5,5).

Endothermic reactions.

(1) At the threshold energy $W_i = \delta$, it is apparent that $a = 0$ $(\gamma_1 = \gamma_2 = \infty)$. Both particles in the final state move in the primary direction, with energies $m_1 \frac{m_I}{M m_{II}} Q$ and $m_2 \frac{m_I}{M m_{II}} Q$ for particles *1* and *2*, respectively.

(2) The quantity γ_1 is smaller than or equal to unity, according as the energy W_i of the incident particle exceeds the threshold energy δ by an amount greater than or equal to

$$\frac{m_1 m_I}{M (m_2 - m_I)} \delta.$$

Hence, the direction of motion of the light particle is limited only when the energy W_i lies in the small interval given by this expression. For $\gamma_1 = 1$, the maximum kinetic energy of particle *1* is about four times greater than the energy which it receives at the threshold.

(3) The quantity γ_2 is smaller than or equal to unity only in reactions in which $m_1 > m_I$ (for instance, $^{14}N\,(n\ \alpha)\ ^{11}B$), and in which the energy W_i exceeds the threshold energy δ by an amount greater than or equal to

$$\frac{m_2 m_I}{M (m_1 - m_I)} \delta.$$

In order to obtain directly the energies of the outgoing particles by a graphical construction, it is convenient to introduce for each particle (*1* and *2*) a separate scale for the drawing of a and α. Thus, we introduce

$$a_1 = \frac{a}{\sqrt{2\,m_1}} = \frac{\sqrt{m_2\,m_{II}}}{M}\sqrt{W_i+\delta'}, \quad c_1 = \frac{\alpha_1}{\sqrt{2\,m_1}} = \frac{\sqrt{m_1\,m_I}}{M}\sqrt{W_i}$$

for particle 1, and

$$a_2 = \frac{a}{\sqrt{2\,m_2}} = \frac{\sqrt{m_1\,m_{II}}}{M}\sqrt{W_i+\delta'}, \quad c_2 = \frac{\alpha_2}{\sqrt{2\,m_2}} = \frac{\sqrt{m_2\,m_I}}{M}\sqrt{W_i}$$

for particle 2. The quantity $(a_n)^2$ is then the energy which particle n receives in the m.c.f. The corresponding vectors in the geometrical picture will then give directly the square roots of the energies of the particles, as is illustrated Fig. 8.

§ 6. Photodisintegration.

In the discussion of photodisintegration, particle I is a photon $(m_I = 0)$ of momentum $\vec{K}$, and particle II is a nucleus of the mass $m_{II} = m$. In the final state, we have a neutron of mass m_1 and the residual nucleus of mass m_2.

For the threshold energy δ of the photon, we obtain from (3,9), (3,6) and (3,7)

$$\delta = |Q|\left(1 + \frac{|Q|}{2\,m}\right).$$

The threshold energy in this case exceeds the binding energy $|Q|$ of the nucleon only by the small amount $\frac{Q^2}{2\,m}$. For the deuteron $(|Q| = 2.2\text{ MeV})$ for example, the threshold for photodisintegration lies only 1.3 keV above the binding energy.

For photon energies small compared with the rest energy of the nucleus $(K \ll m)$, we can confine ourselves, as before, to the non-relativistic approximation, and obtain from (3,8) and (3,9)

$$a = b = \sqrt{\frac{2\,m_1\,m_2}{m}(K-\delta)}$$

and, from (1,12),

$$\alpha_n = \frac{m_n}{m} K.$$

Measuring $K - \delta$ in units of the threshold value δ, i. e. introducing

$$x = \frac{K - \delta}{\delta},$$

we have for the ratios $\gamma_1 = \frac{\alpha_1}{\alpha}$ and $\gamma_2 = \frac{\alpha_2}{\alpha}$:

$$\gamma_1 = \sqrt{\frac{m_1 \delta}{2 m_2 m}} \frac{x+1}{\sqrt{x}}, \quad \gamma_2 = \sqrt{\frac{m_2 \delta}{2 m_1 m}} \frac{x+1}{\sqrt{x}}.$$

These ratios, being infinite at the threshold energy $x = 0$, decrease very rapidly in the neighbourhood of the threshold value and are equal to unity for

$$x_1 \cong \frac{m_1 \delta}{2 m_2 m}, \quad x_2 \cong \frac{m_2 \delta}{2 m_1 m}$$

for the particles *1* and *2*, respectively. Only for the photon energy corresponding to x_1 is it possible for the outgoing neutron to have a vanishing kinetic energy. The photon energy for this case was determined experimentally by WIEDENBECK and MARHOFER [9] for the deuteron (2.185 MeV) and the beryllium nucleus (1.630 MeV). For the deuteron, this energy exceeds the binding energy Q by an amount of ~ 2.5 keV.

A minimum value for γ_1 and γ_2 is reached when $x = 1$, i. e. when $K = 2\delta$. These minimum values of γ_1 and γ_2 are

$$\gamma_{1\,\min} = \sqrt{2 \frac{m_1}{m_2} \frac{\delta}{m}}, \quad \gamma_{2\,\min} = \sqrt{2 \frac{m_2}{m_1} \frac{\delta}{m}}$$

and are very small (due to the smallness of $\frac{\delta}{m}$); therefore, in this case, the two particles are moving nearly in opposite directions.

The three typical cases $K = \delta$, $\gamma_1 = 1$, $K = 2\delta$ are shown, for the deuteron, in Figs. 9a, b, c.

§ 7. Decay of a Meson into Two Particles.

In the initial state, we have a primary particle (meson) of mass $m_I = M$ and momentum $\vec{k}_I = \vec{K}$ $(m_{II} = 0,\ \vec{k}_{II} = 0)$. In the final state resulting from the decay of the primary meson, we have two secondary particles of masses m_1 and m_2. Taking into account that in this process, according to (1,5), $N = M$, we obtain from (1,7) for the semi-minor axis of the momentum ellipsoid (= momentum of the secondary particle if the primary particle were at rest)

$$b = \frac{1}{2M}\sqrt{\{M^2-(m_1+m_2)^2\}\{M^2-(m_1-m_2)^2\}}. \qquad (7,1)$$

The eccentricity of the ellipsoid is equal to the velocity β of the primary particle.

For the following considerations it will be convenient to introduce the abbreviations

$$\mathsf{K} = \frac{K}{M} = \frac{\beta}{\sqrt{1-\beta^2}} \qquad (7,2)$$

$$\mathsf{K}_n^* = \frac{b}{m_n} = \frac{\beta_n^*}{\sqrt{1-\beta_n^{*2}}} \qquad (7,3)$$

where the asterisk refers to the rest system of the primary particle. We can express K_n^* through the kinetic energy w_n^* which the secondary particle receives in the decay of a stopped meson:

$$\mathsf{K}_n^* = \sqrt{\frac{w_n^*}{m_n}\left(2+\frac{w_n^*}{m_n}\right)}. \qquad (7,4)$$

From (1,9) it can be seen that, in the decay of a stopped meson, the kinetic energy w_n^* of the particle *1*, its mass m_1, and the mass ratio $\frac{M}{m_1}$ determine the mass m_2 of the other secondary particle:

$$\frac{m_2}{m_1} = \sqrt{\left(\frac{M}{m_1}-1\right)^2 - 2\frac{M}{m_1}\frac{w_1^*}{m_1}}.$$

The possible angles with which the secondary particle can move relative to the primary direction depend on the ratio

$$\gamma_n = \frac{\alpha_n}{\alpha} = \frac{\beta}{\beta_n^*}.$$

For $\beta > \beta_n^*$, the particle n always forms an acute angle with the primary direction, and the maximum value of this angle is found from (1,21), (7,2), and (7,3) to be:

$$\sin \vartheta_{n\,\max} = \frac{\mathsf{K}_n^*}{\mathsf{K}}. \tag{7,5}$$

Only for $\mathsf{K} \geq \mathsf{K}_n^*$ is the direction of flight limited by (7,5).

The momentum of the particle n emitted in the direction ϑ_n is given by (2,5) or, in our present notation, using (7,2) and (7,3), by the equation

$$\left.\begin{aligned} k_n(\vartheta_n) = \frac{m_n}{1+\mathsf{K}^2 \sin^2 \vartheta_n} \Bigg\{ \mathsf{K} \sqrt{1+\mathsf{K}_n^{*2}} \cos \vartheta_n \pm \\ \pm \mathsf{K}_n^* \sqrt{(1+\mathsf{K}^2)\left(1-\frac{\mathsf{K}^2}{\mathsf{K}_n^{*2}} \sin^2 \vartheta_n\right)} \Bigg\}. \end{aligned}\right\} \tag{7,6}$$

As regards the sign, the same specifications hold as in the case of (2,5), *viz.* for $\mathsf{K} \leq \mathsf{K}_n^*$ we take the $+$ sign, whereas for $\mathsf{K} > \mathsf{K}_n^*$ we have in every direction two possible momenta which differ with regard to sign in (7,6).

In order to observe in the cloud chamber the decay in flight of a meson with a lifetime τ_0, the mean free path for decay

$$l = \frac{\beta c}{\sqrt{1-\beta^2}} \tau_0 = K l_0, \quad l_0 = c \tau_0 \quad (c = \text{light velocity})$$

should be of the order 3—30 cm. For a lifetime $\tau_0 \sim 10^{-10}$ sec ($l_0 \sim 3$ cm) this gives K of the order 1—10, whereas for a lifetime $\tau_0 \sim 10^{-9}$ sec ($l_0 \sim 30$ cm) we must have K of the order 0.1—1.

The value K_n^* for the secondary particle observed in the decay of a π-meson can be estimated from the work of Lattes, Occhialini and Powell [3], who measured the range of the secondary particles arising from the decay of stopped mesons in emulsions. Assuming particular values for the mass of the secondary particle, they obtain corresponding values for its kinetic energy w_n^* from which, by use of (7,4), K_1^* can be determined as follows:

Mass m_1 (in units of the electron mass)	100	150	200	250	300
w_1^* in MeV	3.0	3.6	4.1	4.5	4.85
K_1^*	0.35	0.31	0.29	0.27	0.25

We therefore expect that in all cases where $K > 0.35$, the secondary particle in the laboratory system is emitted within a cone with a half angle given by (7,5).

In the rest system of the primary particle, the secondary particle can be emitted in every direction with the same probability. From the uniform angular distribution in the m.c.f. it follows, as mentioned in § 2, that the probability per unit energy range for the secondary particle having an energy e_n between the limits

$$e_{n\,\min} \leq e_n \leq e_{n\,\max}$$

is constant and equal to

$$\sigma_n(e_n) = \frac{1}{2f} = \frac{M}{2bK}.$$

The minimum and maximum energies of the secondary particle for a given momentum K of the primary particle are given by (2,9).

The angular distribution of the secondary particles is given by (2,12). In the case $K > K_n^*$, the direction of particle n lies within a cone of half angle $\vartheta_{n\,\max}$ and the probability per unit solid angle becomes infinite for $\vartheta_n = \vartheta_{n\,\max}$.

For this case, it is therefore more reasonable to calculate the integral probability $P(\vartheta_n)$ that the direction of the secondary particle lies inside a cone with the half angle ϑ_n. To the angle ϑ_n two angles ϑ_{n+}^* and ϑ_{n-}^* correspond in the m.c.f., and to the angle $\vartheta_n = 0$ correspond in the m.c.f. the angles $\vartheta_n^* = 0$ and $\vartheta_n^* = \pi$. Consequently, the solid angle in the m.c.f., which corresponds to the cone of half angle ϑ_n in the laboratory frame, is

$$2\pi(1-\cos\vartheta^*_{n+})+2\pi(1+\cos\vartheta^*_{n-}).$$

Dividing by 4π, we obtain the probability $P_n(\vartheta_n)$ from (2,2)

$$P(\vartheta_n)=1-\frac{\cos\vartheta_n}{1+\mathsf{K}^2\sin^2\vartheta_n}\bigg/\sqrt{1-\frac{\mathsf{K}^2}{\mathsf{K}_n^{*2}}\sin^2\vartheta_n}.$$

For the decay of an ordinary meson ($M\sim 200\,m_e$) into an electron and a neutrino, we have for the electron, from (7,1) and (7,3),

$$\mathsf{K}_e^*\sim 100.$$

Therefore, the case discussed above, $\mathsf{K}>\mathsf{K}_e^*$, is only realized for $K>100\,M$, i. e. for meson energies above $10^{10}\,eV$. For meson energies much lower than $10^{10}\,eV$ we can neglect $\mathsf{K}^2/\mathsf{K}_e^{*2}$ compared to 1 ($\gamma_e=\beta$) and obtain from (2,12) and (2,5) for the probability per unit solid angle of finding the decay electron at an angle ϑ, φ to the primary direction

$$\sigma(\vartheta,\varphi)=\frac{1}{4\pi}\frac{1-\beta^2}{(1-\beta\cos\vartheta)^2}. \tag{7,7}$$

The probability $P\left(\frac{\pi}{2}\right)$ that the electron goes in a forward direction is, from (7,7),

$$P\left(\frac{\pi}{2}\right)=\frac{1}{2}(1+\beta). \tag{7,8}$$

The formulae (7,7) and (7,8) are exactly valid when the secondary particle under consideration has a vanishing rest mass.

§ 8. β-Decay of Nuclei.

So far we have considered only processes with two free particles in the final state. In the case with three particles in the final state the conservation laws have the form

$$E = \sqrt{m_1^2 + k_1^2} + \sqrt{m_2^2 + k_2^2} + \sqrt{m_3^2 + k_3^2}$$

$$\vec{K} = \vec{k}_1 + \vec{k}_2 + \vec{k}_3 .$$

If we now fix the momentum of one of the three particles, say particle *3*, we have the same situation as in § 1 if, in all equations, $\vec{K}$ and E are replaced by $\vec{K}'$ and E' given by

$$\vec{K}' = \vec{K} - \vec{k}_3$$

$$E' = E - \sqrt{m_3^2 + k_3^2} .$$

The momentum ellipsoid together with the points A_1 and A_2, defined by $\vec{K}'$, E', m_1 and m_2, gives the possible momenta of the particles *1* and *2* for a chosen momentum of particle *3*. It may be noted that the m.c.f. in the preceding considerations is now no more the m.c.f. of the whole system, but only that of the two particles 1 and 2.

We shall now apply the geometrical picture to the β-decay of a nucleus of mass M at rest ($\vec{K} = 0$).[1] In the final state, we have three particles: an electron, a neutrino, and a recoil nucleus. We assume a vanishing rest mass for the neutrino. We shall use the picture for the determination of the possible momenta $\vec{k}_e$ and $\vec{k}_\nu$ of the electron and the neutrino for a given momentum $\vec{k}_r$ of the recoil nucleus. This momentum can vary between the limits $k_{r\,\mathrm{min}} = 0$ and a certain $\vec{k}_{r\,\mathrm{max}}$.

Denoting by m_r and w_r the mass and the kinetic energy of the recoil nucleus, we have

$$\vec{K}' = -\vec{k}_r, \qquad E' = \Delta - w_r, \tag{8,1}$$

where

$$\Delta = M - m_r$$

is the energy difference between the initial and the final nucleus. The energy N' of the system consisting of the electron and the neutrino, in the rest system of these two particles, is now

$$N' = \sqrt{E'^2 - K'^2} = \sqrt{\Delta^2 - 2 M w_r} .$$

1 The possibility of applying the geometrical picture to the problem of β-decay was suggested by mag. sc. O. Kofoed-Hansen.

The semi-minor axis of the momentum ellipsoid is then, according to (1,7),

$$b = \frac{\Delta^2 - m_e^2 - 2\,M\,w_r}{2\,\sqrt{\Delta^2 - 2\,M\,w_r}}. \tag{8,2}$$

Since b cannot be negative, we see that the maximum value of w_r is given by

$$w_{r\,\max} = \frac{\Delta^2 - m_e^2}{2\,M}$$

and, consequently, Δ can be determined from a measurement of the maximum recoil energy [10].

For this maximum value of w_r, we have $b = 0$ and, therefore, $\alpha_\nu = f = 0$ (α_ν referring to the neutrino) so that the neutrino energy vanishes. Since the kinetic energy of the nucleus is very small we can put, with a sufficient degree of approximation,

$$k_r^2 = 2\,M\,w_r$$

thus obtaining from (8,2)

$$b = \frac{k_{r\,\max}^2 - k_r^2}{2\,\sqrt{\Delta^2 - k_r^2}}.$$

For the eccentricity of the ellipsoid we have, from (1,6) and (8,1),

$$\beta = \frac{k_r}{\Delta} \tag{8,3}$$

and for the semi-major axis, in the same approximation,

$$a = \Delta\,\frac{k_{r\,\max}^2 - k_r^2}{2\,(\Delta^2 - k_r^2)}.$$

The momentum ellipsoid goes for $k_r = 0$ over in a sphere and reduces for $k_{r\,\max}$ to a point. The semi-major and semi-minor axes decrease with increasing k_r.

The origin A_e of the momentum vector of the electron is at the distance α_e apart from the centre of the ellipsoid

$$\alpha_e = \frac{k_r}{2}\left(1 + \frac{m_e^2}{\Delta^2 - k_r^2}\right), \tag{8,4}$$

while α_ν is given by

$$\alpha_\nu = f = \frac{k_r M}{\left(\Delta^2 - k_r^2\right)}\left(w_{r\,\max} - w_r\right) = \frac{k_r\left(k_{r\,\max}^2 - k_r^2\right)}{2\left(\Delta^2 - k_r^2\right)}. \tag{8,5}$$

The lower and upper limits of possible kinetic energies of the electron, for a given momentum of the recoil nucleus, are from (2,9) equal to

$$w_{e\,\min} = \frac{\alpha_e}{\beta} - m_e - f, \quad w_{e\,\max} = \frac{\alpha_e}{\beta} - m_e + f,$$

respectively. Putting for α_e, β and f the values given by (8,3), (8,4), and (8,5), we obtain

$$w_{e\,\min} = \frac{\left(\Delta - m_e - k_r\right)^2}{2\left(\Delta - k_r\right)}, \quad w_{e\,\max} = \frac{\left(\Delta - m_e + k_r\right)^2}{2\left(\Delta + k_r\right)}.$$

In carrying out integrations over the possible energies of the electron for a given energy of the recoil nucleus, it may be noticed that w_e can assume all values between $w_{e\,\min}$ and $w_{e\,\max}$ and that, since the point A_e lies outside the focus of the momentum ellipsoid, the angle ϑ_e is a single valued function of w_e.

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REFERENCES

1. L. Landau and E. Lifschitz, Theoretical Physics, vol. IV, Moscow 1941.
2. G. Wataghin, Acad. Bras. Sciencias, 1941.
3. C. M. G. Lattes, G. P. S. Occhialini, F. C. Powell, Nature **160**, 453, 486, 1947.
4. C. E. Mandeville, Journal of the Franklin Institute **244**, 385, 1947.
5. A. Rogozinski, Phys. Rev. **65**, 300, 1944.
6. L. Leprince Ringuet, S. Gorodetzky, E. Nageotte and R. Richard-Foy, Phys. Rev. **59**, 460, 1941.
7. F. C. Champion, Proc. Roy. Soc. A **136**, 630, 1932.
8. N. N. Das Gupta and S. K. Ghosh, Rev. Mod. Phys. **18**, 225, 1946.
9. M. L. Wiedenbeck and C. J. Marhoefer, Phys. Rev. **67**, 54, 1945.
10. J. C. Jacobsen and O. Kofoed-Hansen, Det Kgl. Danske Vidensk. Selskab, Mat.-fys. Medd. **23**, Nr. 12, 1945.

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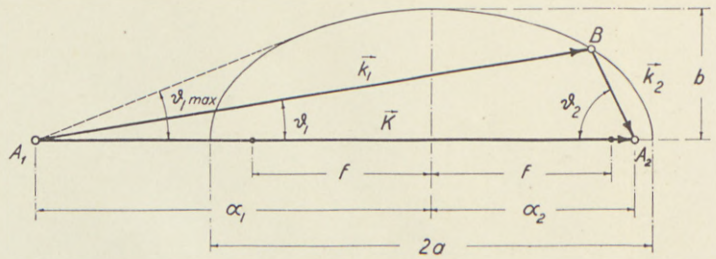

Fig. 1. Geometrical picture of energy-momentum conservation. The semi-minor axis of the ellipse is given by (1,7). The eccentricity is equal to the mass centre velocity of the system. $\vec{k}_1$, $\vec{k}_2$ are momenta of the particles in the final state.

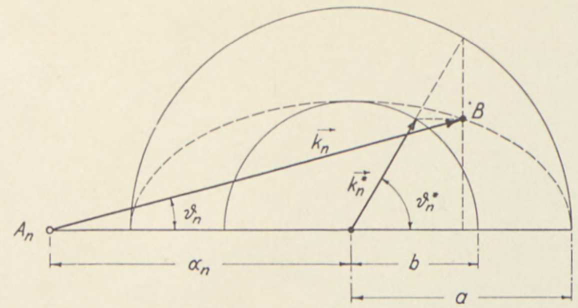

Fig. 3. Transition from the m.c.f. to the laboratory frame of reference. Quantities referring to the m.c.f. are denoted by an asterisk. The vector $\vec{k}_n^*$ with the origin in the centre of the ellipsoid has the length b and its component perpendicular to $\vec{K}$ is equal to the component of the vector $\vec{k}_n$ in the same direction, since $\vec{k}_n^*$ and $\vec{k}_n$ are connected by a Lorentz transformation with a relative velocity in the direction of $\vec{K}$.

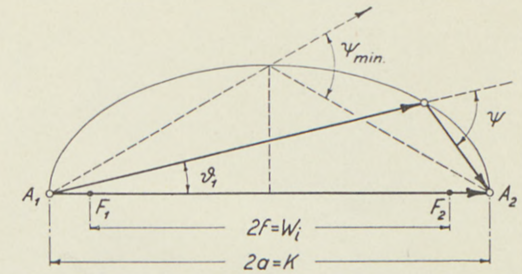

Fig. 4. Elastic collision between two particles of equal mass. The velocities after the collision form always an acute angle $\psi = \vartheta_1 + \vartheta_2$ which is larger than or equal to $\psi_{\min}\left(\cos\psi_{\min} = \frac{W_i}{W_i + 4m}\right)$. The distance $2f$ between the foci = kinetic energy W_i of the incident particle, and the major axis $2a = K$.

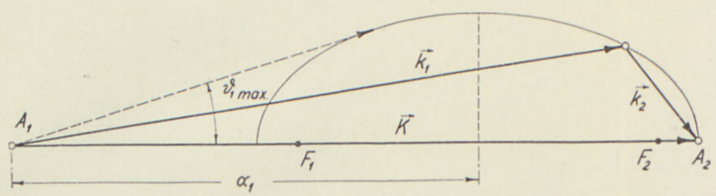

Fig. 5. Elastic collision in which the incident particle has a larger mass than the target particle ($m_1 > m_2$). Maximum deflection angle of the incident particle is given by $\sin\vartheta_{1\,\max} = \frac{m_2}{m_1}$. For instance, collision of a meson with an electron.

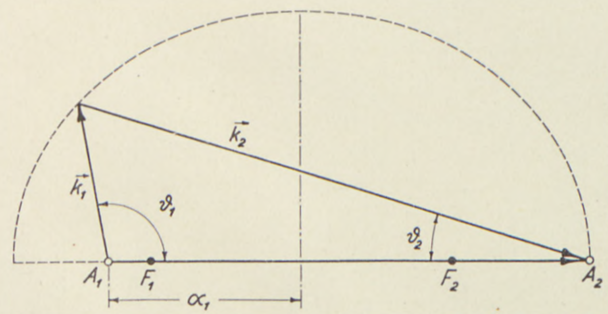

Fig. 6. Elastic collision in which the incident particle has a smaller rest mass than the target particle, for instance, scattering of photons by free electrons (in this case, A_1 coincides with F_1).

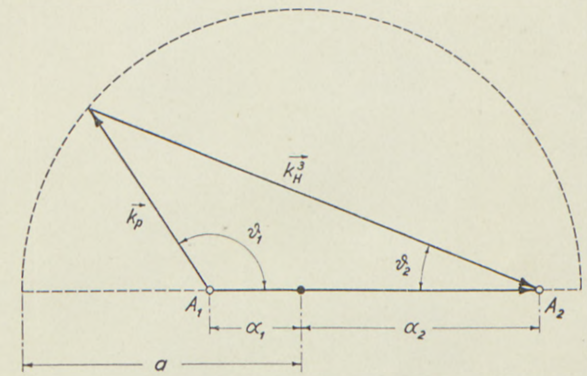

Fig. 7. Geometrical picture of the exothermic reaction $^2D\,(d, p)\,^3H$, ($Q \simeq 4\,MeV$) for an energy $W_i = 2\,MeV$ of the incident deuteron. $a = 83.5\,MeV$, $a_1 = 21.5\,MeV$, $a_2 = 3a_1$.

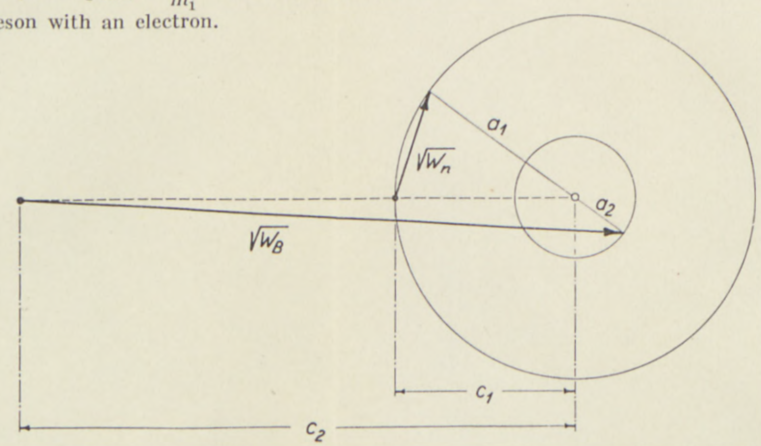

Fig. 8. Picture giving the energies of the outcoming particles in the endothermic reaction

$^9Be(p,n)^9B$ ($Q = -1{,}83\,MeV$, $\delta = 2{,}03\,MeV$, $a_1 = \frac{9}{10}\sqrt{W_i - 2{,}03}$, $a_2 = \frac{1}{3}a_1$, $c_1 = \frac{1}{10}\sqrt{W_i}$, $c_2 = 3c_1$)

for an energy W_i which is $25\,keV$ above the threshold where $a_1 = c_1 = 4.5\sqrt{keV}$. w_n and w_B represent the energies of the outgoing neutron and boron nucleus.

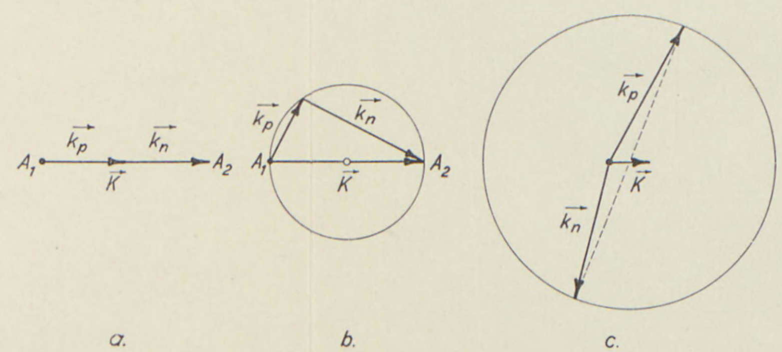

Fig. 9. Photodisintegration of the deuteron. a. $K = \delta$, $a = 0$. Exactly at the threshold ($1.3\,keV$ above the binding energy of the deuteron), the proton and the neutron are moving in the primary direction and have each an energy $0{,}65\,keV$. b. $K = \delta\left(1 + \frac{\delta}{2m}\right)$, $\gamma = 1$. At an energy $1.3\,keV$ above the threshold, the proton and the neutron paths form a right angle whatever their angle with the primary direction. c. $K = 2\delta$, $\gamma = 0.04$, the proton and the neutron are moving in nearly opposite directions.